Discontinuous Galerkin Methods for Porous-Media Flow with DUNE-FEM

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Table of contents

Motivation

DG methods for numerical simulations of complex geological systems

- Pros
    - High order convergence (depending on regularity)
    - Local conservation of physical quantities such as mass, momentum, and energy
    - Nonmatching grids, hp-adaptivity
    - Efficient use of memory hierarchy due to dense blocks
- Cons
    - Large number of degrees of freedom
    - ill-conditioning and denser global matrix with increasing approximation order

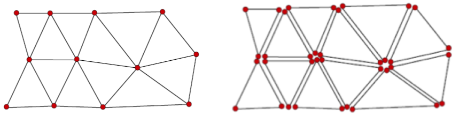
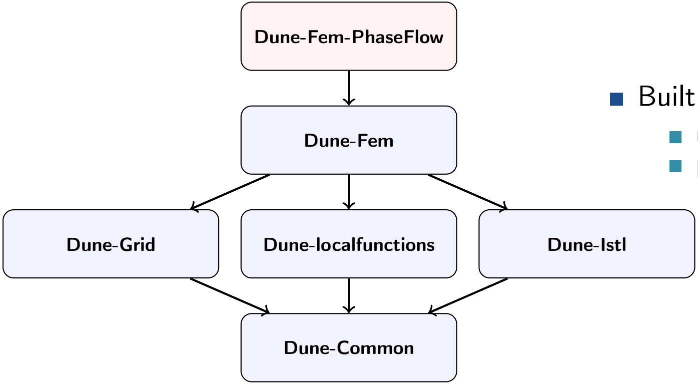


Abbildung : CG vs DG (dofs for piecewise linear)

Motivation

Challenge: Develop efficients implementation with DG on a sustainable framework.



- Built on top of Dune-Fem
  - user-friendly implementation,
  - profit from many features,
    - DG discretization spaces,
    - efficient solvers and grids,
    - support for parallelization and adaptivity.

Dune FEM The FEM Module ! Distributed and Unified Numerics Environment

Dune

Model formulation

Domain $\Omega \in \mathbb{R}^d$, $d \in \{1,2,3\}$.

- Phases={w,n}
  - both phases incompressible,
  - no dissolution.
- Model might include gravity,
- consider media heterogeneities.

s_n-p_w formulation

Two coupled equations for p_w; s_n :

$$
\begin{aligned}
-\nabla\cdot[(\lambda_w+\lambda_n)K\nabla p_w+\lambda_n K\nabla p_c-(\rho_w\lambda_w+\rho_n\lambda_n)Kg]&=q_w+q_n,\\
\phi\frac{\partial s_n}{\partial t}-\nabla\cdot[\lambda_n K(\nabla p_w-\rho_n g)]-\nabla\cdot[\lambda_n K\nabla p_c]&=q_n.
\end{aligned}
\tag{1}
$$

Here ϕ is the porosity, K is the permeability and q_w, q_n are source/sink term.

Non linearities

- Capillary pressure $p_c = p_c(s_n)$,
- Phases mobilities $\lambda_w = \lambda_w(s_n) = \frac{k_{rw}(s_n)}{\mu_w}$, $\lambda_n = \lambda_n(s_n) = \frac{k_{rn}(s_n)}{\mu_n}$

  where μ_α is the viscosity and $k_{r\alpha}$ is the relative permeability of phase $\alpha = \{w,n\}$.

Boundary conditions & initial values

Boundary divided into disjoint open sets $\partial\Omega = \Gamma^p_{w_{Dir}} \cup \Gamma^s_{n_{Dir}} \cup \Gamma_{\alpha_{Neum}}$.

Boundary & initial conditions

$$p_c(x,0) = p_{c_{Dir}}(0) \text{ or } s_n(x,0) = s_n^0(x),\ p_w(x,0) = p_w^0(x) \qquad \forall x \in \Omega, \tag{2}$$

$$p_c(x,t) = p_{c_{Dir}}(x,t) \text{ or } s_n(x,t) = s_{n_{Dir}}(x,t) \qquad \forall x \in \Gamma^s_{n_{Dir}}, \tag{3}$$

$$p_w(x,t) = p_{w_{Dir}}(x,t) \qquad \forall x \in \Gamma^p_{w_{Dir}}, \tag{4}$$

$$\rho_\alpha u_\alpha \cdot n = J_\alpha(x,t) \text{ and } J_t = \sum_{\alpha\in\{w,n\}} J_\alpha \qquad \forall x \in \Gamma_{\alpha_{Neum}}. \tag{5}$$

Here n the outward normal to $\partial\Omega$ and $J_\alpha,\ \alpha \in \{w,n\}$ is the inflow.

DG Finite element space

Domain Ω is subdivided into a partition $\mathscr{T}_h = \{E\}$ consisting of N_h elements.

DG Finite element space

The discontinuous finite element space is:
$\mathscr{D}_r(\mathscr{T}_h) = \{v \in \mathbb{L}^2(\Omega) : v_{|E} \in \mathbb{P}_r(E) \ \ \forall E \in \mathscr{T}_h\}$, with $\mathbb{P}_r(E)$ the space of polynomial functions of degree at most $1 \le r$ on E.

- r_p for the pressure,
- r_s for the saturation.

```
 //Legendre DG space
   typedef Dune::Fem::LegendreDiscontinuousGalerkinSpace<
   FunctionSpaceType, GridPartType, POLORDER >
   DiscreteFunctionSpaceType;
```

```
//Lagrange DG space
  typedef Dune::Fem::LagrangeDiscontinuousGalerkinSpace<
   FunctionSpaceType, GridPartType, POLORDER >
   DiscreteFunctionSpaceType;
```

Jump & Weighted average operator

Different types of domain can meet closely and cause large jumps in permeability.

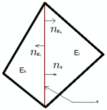


Abbildung : Two neigboring cells

Abbildung : Average & Jump

Jump & Weighted average operators

The jump is:

$$[\![p]\!] = p_{E_1} n_{E_1} + p_{E_2} n_{E_2}. \tag{6}$$

The weighted average:

$$\{p\}_\omega = \omega_{E_1} p_{E_1} + \omega_{E_2} p_{E_2}. \tag{7}$$

Here $\omega_{E_1} = \frac{\delta_K^{E1}}{\delta_K^{E1} + \delta_K^{E2}}$ and $\omega_{E_2} = \frac{\delta_K^{E2}}{\delta_K^{E1} + \delta_K^{E2}}$ with $\delta_K^{E1} = n_e^T K_{E_2} n_e$ and $\delta_K^{E2} = n_e^T K_{E_1} n_e$, K_{E_1} and K_{E_2} are the absolute permeabilities for E_1 and E_2.

Phase conservation equation

The saturation equation is: $\phi \frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_n K (\nabla p_w - \rho_n g)) - \nabla \cdot (\lambda_n K \nabla p_c) = q_n$.

Phase conservation equation

The saturation equation is: $\phi \frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_n K(\nabla p_w - \rho_n g)) - \nabla \cdot (\lambda_n K \nabla p_c) = q_n$.

Non wetting phase conservation

$$\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i)z$$

accumulation term

Phase conservation equation

The saturation equation is: $\phi \frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_n K(\nabla p_w - \rho_n g)) - \nabla \cdot (\lambda_n K \nabla p_c) = q_n$.

Non wetting phase conservation

$$\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i) z + \sum_{E \in \mathcal{T}_h} \int_E (K \lambda_n^{i+1} \nabla p_w^{i+1} + \lambda_n^{i+1} K \nabla p_c^{i+1} - \rho_n \lambda_n^{i+1} K g) \cdot \nabla z$$

volume contribution

Phase conservation equation

The saturation equation is: $\phi\frac{\partial s_n}{\partial t} - \nabla\cdot(\lambda_n K(\nabla p_w - \rho_n g)) - \nabla\cdot(\lambda_n K \nabla p_c) = q_n.$

Non wetting phase conservation

$$\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i)z + \sum_{E\in\mathcal{T}_h}\int_E (K\lambda_n^{i+1}\nabla p_w^{i+1} + \lambda_n^{i+1}K\nabla p_c^{i+1} - \rho_n\lambda_n^{i+1}Kg)\cdot\nabla z$$

consistency term

$$- \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow,i+1}\{K\nabla p_w^{i+1}\cdot n_e\}_\omega [\![z]\!] + \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \lambda_n^{\uparrow,i+1}\{\rho_n Kg)\cdot n_e\}_\omega [\![z]\!]$$

$$- \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1}K\nabla p_c^{i+1}\cdot n_e\}_\omega [\![z]\!]$$

Phase conservation equation

The saturation equation is: $\phi\frac{\partial s_n}{\partial t} - \nabla\cdot(\lambda_n K(\nabla p_w - \rho_n g)) - \nabla\cdot(\lambda_n K \nabla p_c) = q_n.$

Non wetting phase conservation

$$\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i) z + \sum_{E\in\mathcal{T}_h}\int_E (K\lambda_n^{i+1}\nabla p_w^{i+1} + \lambda_n^{i+1} K \nabla p_c^{i+1} - \rho_n \lambda_n^{i+1} K g)\cdot\nabla z$$

$$- \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow,i+1}\{K\nabla p_w^{i+1}\cdot n_e\}_\omega [\![z]\!] + \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \lambda_n^{\uparrow,i+1}\{\rho_n K g)\cdot n_e\}_\omega [\![z]\!]$$

$$- \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1} K \nabla p_c^{i+1}\cdot n_e\}_\omega [\![z]\!] + \epsilon \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1} K\nabla z\cdot n_e\}_\omega [\![p_c^{i+1}]\!]$$

$$+ \epsilon \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow,i+1}\{K\nabla z\cdot n_e\}_\omega [\![p_w^{i+1}]\!] +$$

symetry term

Phase conservation equation

The saturation equation is: $\phi \frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_n K(\nabla p_w - \rho_n g)) - \nabla \cdot (\lambda_n K \nabla p_c) = q_n.$

Non wetting phase conservation

$$\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i) z + \sum_{E \in \mathcal{T}_h} \int_E (K \lambda_n^{i+1} \nabla p_w^{i+1} + \lambda_n^{i+1} K \nabla p_c^{i+1} - \rho_n \lambda_n^{i+1} K g) \cdot \nabla z$$

$$- \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D} \cup \Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow, i+1} \{K \nabla p_w^{i+1} \cdot n_e\}_\omega [\![z]\!] + \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \lambda_n^{\uparrow, i+1} \{\rho_n K g) \cdot n_e\}_\omega [\![z]\!]$$

$$- \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1} K \nabla p_c^{i+1} \cdot n_e\}_\omega [\![z]\!] + \epsilon \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1} K \nabla z \cdot n_e\}_\omega [\![p_c^{i+1}]\!]$$

$$+ \epsilon \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D} \cup \Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow, i+1} \{K \nabla z \cdot n_e\}_\omega [\![p_w^{i+1}]\!] + \boxed{\sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \gamma_e^s \int_e [\![p_c^{i+1}]\!] [\![z]\!]}$$

stability term

Phase conservation equation

The saturation equation is: $\phi \frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_n K (\nabla p_w - \rho_n g)) - \nabla \cdot (\lambda_n K \nabla p_c) = q_n$.

Non wetting phase conservation

$$
\begin{aligned}
&\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i) z + \sum_{E \in \mathcal{T}_h} \int_E (K \lambda_n^{i+1} \nabla p_w^{i+1} + \lambda_n^{i+1} K \nabla p_c^{i+1} - \rho_n \lambda_n^{i+1} K g) \cdot \nabla z \\
&- \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D} \cup \Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow, i+1} \{K \nabla p_w^{i+1} \cdot n_e\}_\omega [\![z]\!] + \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \lambda_n^{\uparrow, i+1} \{\rho_n K g) \cdot n_e\}_\omega [\![z]\!] \\
&- \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1} K \nabla p_c^{i+1} \cdot n_e\}_\omega [\![z]\!] + \epsilon \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1} K \nabla z \cdot n_e\}_\omega [\![p_c^{i+1}]\!] \\
&+ \epsilon \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D} \cup \Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow, i+1} \{K \nabla z \cdot n_e\}_\omega [\![p_w^{i+1}]\!] + \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \gamma_e^s \int_e [\![p_c^{i+1}]\!][\![z]\!] \\
&= \int_\Omega q_n z - \sum_{e \in \Gamma_{n_N}} \int_e J_n^{i+1} z + \epsilon \sum_{e \in \Gamma^h_{s_D} \cup \Gamma^h_{p_D}} \int_e \lambda_n^{i+1} K \nabla z \cdot n_e p_D \\
&+ \epsilon \sum_{e \in \Gamma^h_{s_D}} \int_e \lambda_n^{i+1} K \nabla z \cdot n_e p_c(s_D) + \sum_{e \in \Gamma^h_{s_D}} \gamma_e^s \int_e p_c(s_D) z, \qquad \forall z \in \mathcal{D}_{r_s}(\mathcal{T}_h)
\end{aligned}
\tag{8}
$$

rhs

Phase conservation equation

The saturation equation is: $\phi \frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_n K(\nabla p_w - \rho_n g)) - \nabla \cdot (\lambda_n K \nabla p_c) = q_n.$

Non wetting phase conservation

$$
\begin{aligned}
&\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i)z + \sum_{E \in \mathcal{T}_h} \int_E (K\lambda_n^{i+1}\nabla p_w^{i+1} + \lambda_n^{i+1} K \nabla p_c^{i+1} - \rho_n \lambda_n^{i+1} K g) \cdot \nabla z \\
&- \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D} \cup \Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow, i+1} \{K \nabla p_w^{i+1} \cdot n_e\}_\omega [\![z]\!] + \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \lambda_n^{\uparrow, i+1} \{\rho_n K g) \cdot n_e\}_\omega [\![z]\!] \\
&- \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1} K \nabla p_c^{i+1} \cdot n_e\}_\omega [\![z]\!] + \epsilon \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1} K \nabla z \cdot n_e\}_\omega [\![p_c^{i+1}]\!] \\
&+ \epsilon \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D} \cup \Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow, i+1} \{K \nabla z \cdot n_e\}_\omega [\![p_w^{i+1}]\!] + \sum_{e \in \Gamma^h \cup \Gamma^h_{s_D}} \gamma_e^s \int_e [\![p_c^{i+1}]\!][\![z]\!] \\
&= \int_\Omega q_n z - \sum_{e \in \Gamma_{n_N}} \int_e J_n^{i+1} z + \epsilon \sum_{e \in \Gamma^h_{s_D} \cup \Gamma^h_{p_D}} \int_e \lambda_n^{i+1} K \nabla z \cdot n_e p_D \\
&+ \epsilon \sum_{e \in \Gamma^h_{s_D}} \int_e \lambda_n^{i+1} K \nabla z \cdot n_e p_c(s_D) + \sum_{e \in \Gamma^h_{s_D}} \gamma_e^s \int_e p_c(s_D) z, \qquad \forall z \in \mathcal{D}_{r_s}(\mathcal{T}_h)
\end{aligned}
\tag{8}
$$

Upwinding of mobility:

$$
\lambda_n^{\uparrow, i+1} = \begin{cases} \lambda_{n,E_2}^{i+1} & \text{if } -K(\nabla p_w - \nabla p_c - \rho_n g) \cdot n \geq 0 \\ \lambda_{n,E_1}^{i+1} & \text{else} \end{cases}
$$

Phase conservation equation

The saturation equation is: $\phi \frac{\partial s_n}{\partial t} - \nabla\cdot(\lambda_n K(\nabla p_w - \rho_n g)) - \nabla\cdot(\lambda_n K \nabla p_c) = q_n.$

Non wetting phase conservation

$$
\begin{aligned}
&\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i)z + \sum_{E\in\mathcal{T}_h} \int_E (K\lambda_n^{i+1}\nabla p_w^{i+1} + \lambda_n^{i+1}K\nabla p_c^{i+1} - \rho_n\lambda_n^{i+1}Kg)\cdot\nabla z \\
&- \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow,i+1}\{K\nabla p_w^{i+1}\cdot n_e\}_\omega [\![z]\!] + \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \lambda_n^{\uparrow,i+1}\{\rho_n Kg)\cdot n_e\}_\omega [\![z]\!] \\
&- \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1}K\nabla p_c^{i+1}\cdot n_e\}_\omega [\![z]\!] + \epsilon \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1}K\nabla z\cdot n_e\}_\omega [\![p_c^{i+1}]\!] \\
&+ \epsilon \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow,i+1}\{K\nabla z\cdot n_e\}_\omega [\![p_w^{i+1}]\!] + \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \gamma_e^s \int_e [\![p_c^{i+1}]\!][\![z]\!] \\
&= \int_\Omega q_n z - \sum_{e\in\Gamma_{n_N}} \int_e J_n^{i+1} z + \epsilon \sum_{e\in\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{i+1} K\nabla z\cdot n_e p_D \\
&+ \epsilon \sum_{e\in\Gamma^h_{s_D}} \int_e \lambda_n^{i+1} K\nabla z\cdot n_e p_c(s_D) + \sum_{e\in\Gamma^h_{s_D}} \gamma_e^s \int_e p_c(s_D) z, \qquad \forall z\in\mathcal{D}_{r_s}(\mathcal{T}_h)
\end{aligned}
\tag{8}
$$

Different DG methods:

$$
\epsilon = \begin{cases} -1 & \text{SIPG} \\ 1 & \text{NIPG} \\ 0 & \text{IIPG} \end{cases}
$$

Phase conservation equation

The saturation equation is: $\phi\frac{\partial s_n}{\partial t} - \nabla\cdot(\lambda_n K(\nabla p_w - \rho_n g)) - \nabla\cdot(\lambda_n K\nabla p_c) = q_n$.

Non wetting phase conservation

$$\begin{aligned}
&\int_\Omega \frac{\phi}{\Delta t}(s_n^{i+1} - s_n^i)z + \sum_{E\in\mathcal{T}_h}\int_E (K\lambda_n^{i+1}\nabla p_w^{i+1} + \lambda_n^{i+1}K\nabla p_c^{i+1} - \rho_n\lambda_n^{i+1}Kg)\cdot\nabla z \\
&- \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow,i+1}\{K\nabla p_w^{i+1}\cdot n_e\}_\omega [\![z]\!] + \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \lambda_n^{\uparrow,i+1}\{\rho_n Kg)\cdot n_e\}_\omega [\![z]\!] \\
&- \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1}K\nabla p_c^{i+1}\cdot n_e\}_\omega [\![z]\!] + \epsilon \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \int_e \{\lambda_n^{i+1}K\nabla z\cdot n_e\}_\omega [\![p_c^{i+1}]\!] \\
&+ \epsilon \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{\uparrow,i+1}\{K\nabla z\cdot n_e\}_\omega [\![p_w^{i+1}]\!] + \sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}} \gamma_e^s \int_e [\![p_c^{i+1}]\!][\![z]\!] \\
&= \int_\Omega q_n z - \sum_{e\in\Gamma_{n_N}} \int_e J_n^{i+1} z + \epsilon \sum_{e\in\Gamma^h_{s_D}\cup\Gamma^h_{p_D}} \int_e \lambda_n^{i+1}K\nabla z\cdot n_e p_D \\
&+ \epsilon \sum_{e\in\Gamma^h_{s_D}} \int_e \lambda_n^{i+1}K\nabla z\cdot n_e p_c(s_D) + \sum_{e\in\Gamma^h_{s_D}} \gamma_e^s \int_e p_c(s_D)z, \qquad \forall z\in\mathcal{D}_{r_s}(\mathcal{T}_h)
\end{aligned} \tag{8}$$

Choice of the penalty term is crucial:

$$\gamma_e^s = C(K_{E_1}, K_{E_2}, P_{eE_1}, P_{eE_2})\frac{p(p+d-1)|e|}{min(|E_1|,|E_2|)}.$$

Total fluid conservation equation

$$-\nabla\cdot(\lambda_t K\nabla p_w+\lambda_n K\nabla p_c-(\rho_w\lambda_w+\rho_n\lambda_n)Kg)=q_w+q_n$$

Total fluid conservation

$$\begin{aligned}
&\sum_{E\in\mathcal{T}_h}\int_E(\lambda_t^{i+1}K\nabla p_w^{i+1}+\lambda_n^{i+1}K\nabla p_c^{i+1}-(\rho_n\lambda_n^{i+1}+\rho_w\lambda_w^{i+1})Kg)\cdot\nabla v\\
&-\sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}}\int_e\{\lambda_t^{i+1}K\nabla p_w^{i+1}\cdot n_e\}_\omega[\![v]\!]\\
&-\sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}}\int_e\{\lambda_n^{i+1}K\nabla p_c^{i+1}\cdot n_e\}_\omega[\![v]\!]+\sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}}\int_e\{(\rho_n\lambda_n^{i+1}+\rho_w\lambda_w^{i+1})Kg)\cdot n_e\}_\omega[\![v]\!]\\
&+\epsilon\sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}\cup\Gamma^h_{p_D}}\int_e\{\lambda_t^{i+1}K\nabla v\cdot n_e\}_\omega[\![p_w^{i+1}]\!]\\
&+\epsilon\sum_{e\in\Gamma^h\cup\Gamma^h_{s_D}}\int_e\{\lambda_n^{i+1}K\nabla v\cdot n_e\}_\omega[\![p_c^{i+1}]\!]+\sum_{e\in\Gamma^h\cup\Gamma^h_{p_D}}\gamma_e^p\int_e[\![p_w^{i+1}]\!][\![v]\!]\\
&=\int_\Omega(q_w^{i+1}+q_n^{i+1})v-\sum_{e\in\Gamma_{w_N}\cup\Gamma_{n_N}}\int_e J_t^{i+1}v\\
&+\epsilon\sum_{e\in\Gamma^h_{s_D}\cup\Gamma^h_{p_D}}\int_e\lambda_t^{i+1}K\nabla v\cdot n_e p_D+\epsilon\sum_{e\in\Gamma^h_{s_D}}\int_e\lambda_n^{i+1}K\nabla v\cdot n_e p_c(s_D)\\
&+\sum_{e\in\Gamma^h_{p_D}}\gamma_e^p\int_e p_D v\qquad\forall v\in\mathcal{D}_{r_p}(\mathcal{T}_h)
\end{aligned}\tag{9}$$

System of non-linear algebraic equations

System of non-linear algebraic equations

At each Newton step, we solve the coupled linear system:

$$\begin{pmatrix} J_{pp} & J_{ps} \\ J_{sp} & J_{ss} \end{pmatrix} \begin{pmatrix} x_p \\ x_s \end{pmatrix} = \begin{pmatrix} b_p \\ b_s \end{pmatrix}$$

```
   //! define Jacobian operator
  typedef DifferentiableFlowOperator< LinearOperatorType,
   ModelType > FlowOperatorType;
```

```
   //! define Newton Inverse Operator
typedef Dune::Fem::NewtonInverseOperator< typename
   BaseType::LinearOperatorType,
    typename BaseType::LinearInverseOperatorType >
    NonLinearInverseOperatorType;
```

Solve the linear systems

Implement solvers as "inverse" operators

- DUNE-ISTL
  - BiCG-stab solver

```
//! define ISTL BiCG-stab operator
typedef Dune::Fem::ISTLBICGSTABOp< DiscreteFunctionType
, LinearOperatorType > LinearInverseOperatorType;
```

  - GMRES solver (see class ISTLGMResOp in dune/fem/solver/istlsolver.hh)
- PetSc
  - FGMRES
  - CGR

Exploiting (and preserving) these algebraic properties of each individual block facilitates the development of robust preconditioners.

- Jacobi
- AMG
- ILU-0

EOC for a Benchmark problem

We consider a system of partial differential equations with known exact solution.

- compute the EOC

Problem

Considering $\Omega = (0,1)^2$ and $J = (0,T)$, find (p,s) such that

$$-\nabla\cdot(\lambda(s)K\nabla p) = 0 \qquad \text{in } \Omega\times J \tag{10}$$

$$\phi\frac{\partial s}{\partial t} - \nabla\cdot(-\epsilon\nabla s + f(s)\lambda(s)K\nabla p) = q \qquad \text{in } \Omega\times J \tag{11}$$

with $\lambda(s) = (0.5-0.2s)^{-1}$, $\epsilon = 0.01$, $f(s) = s$, where $q = 2\pi\epsilon\sin(\pi(x_1+x_2-2t))$

Boundary & Initial conditions

$$p(x,y,t) = \frac{0.2}{\pi}\cos(\pi(x+y-2t)) - 0.5(x+y), \quad s(x,y,t) = \sin(\pi(x+y-2t)) \quad \forall\,(x,y,t)\in\partial\Omega\times J$$

$$p(x,y,t) = \frac{0.2}{\pi}\cos(\pi(x+y-2t)) - 0.5(x+y), \quad s(x,y,t) = \sin(\pi(x+y-2t)) \quad \forall\,(x,y,t)\in\Omega$$

EOC for a Benchmark problem

- Pressure and Saturation are approximated by piecewise polynomials of order k, $k \in \{1,2,3\}$
- no upwinding, no slope limiting techniques
- no $\mathbb{H}(div)$ reconstruction of total velocity.
- Penalty parameter $\sigma = 10$ as in [ref MOzo].
- Time step of $3.125e-4$

The errors and convergence orders of DG are presented for pressure and saturation at final time T = 0.2.

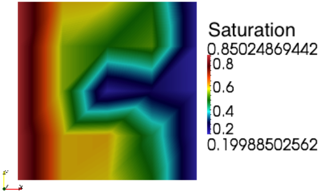
k	$\|p-p_h\|_{L^2(\Omega)}$	EOC
1	$3.11\ 10^{-3}$	1.70207
2	$6.836\ 10^{-4}$	2.9021
3	$6.46266e-05$	3.853

Tabelle : L^2 error and convergence orders of DG method for pressure at T=0.2.

k	$\|s-s_h\|_{L^2(\Omega)}$	EOC
1	$3.32\ 10^{-2}$	2.04561
2	$4.29\ 10^{-4}$	2.823
3	$2.44\ 10^{-4}$	3.870

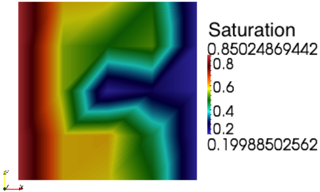
Tabelle : L^2 error and convergence orders of DG method for saturation at $T = 0.2$.

Strongly heterogeneous porous media

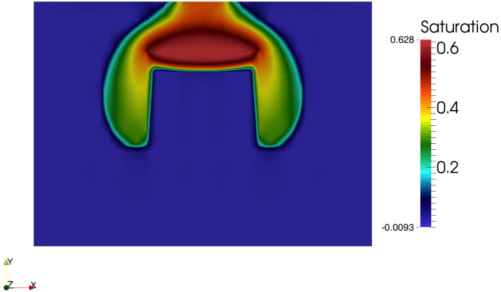


2d Horizontal heterogeneous flow: Saturation distribution after 550 days. Polynomials of order $k = 1$ Domain permeability 1e-8 m^2, lens permeability 1e-12 m^2. Width=100m, depth=100m.

Strongly heterogeneous porous media



2d Horizontal heterogeneous flow: Saturation distribution after 550 days. Polynomials of order $k = 1$ Domain permeability 1e-8 m^2, lens permeability 1e-12 m^2. Width=100m, depth=100m.



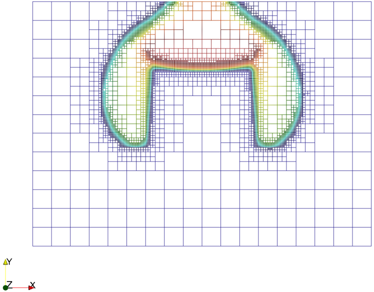
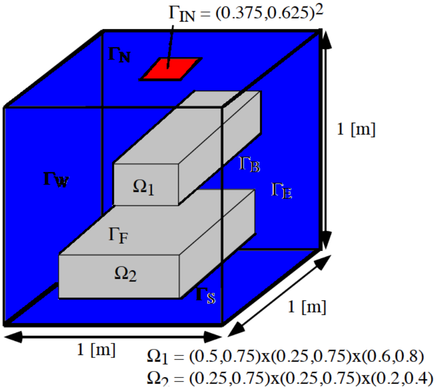


Abbildung : 2d-Problem: Saturation distribution after 2650 s of injection with 0.075 kg/s of DNAPL in a depht of 0.65 m Domain permeability 6e-11 m^2. Lens permeability 7e-13 m^2

Strongly heterogeneous porous media



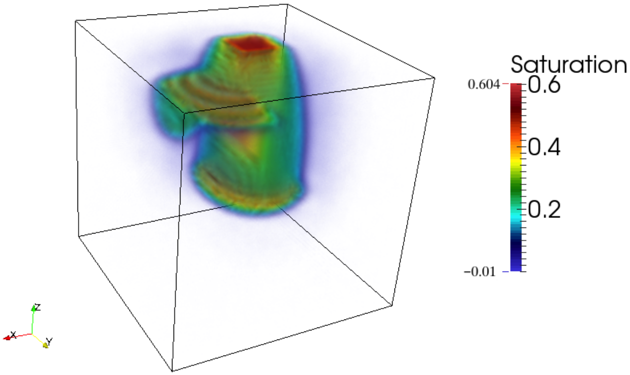


Abbildung : 3d-Problem: Saturation distribution after 5000 s of injection with 0.25 kg/s of DNAPL in a depht of 1m Domain permeability 6e-11 m^2. Lenses permeabilities 7.15e-13 m^2, 7.15e-14 m^2

Thank you for your interest!