Hybridized Discontinuous Galerkin Methods
Theory and Implementation in DUNE

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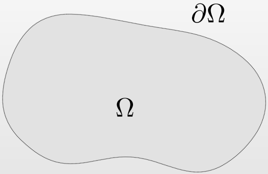
Joint work with Herbert Egger (Uni Graz)

1st DUNE User Meeting, Stuttgart

Outline

Outline

Model problem



Model problem: Poisson's equation

Given $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$, such that

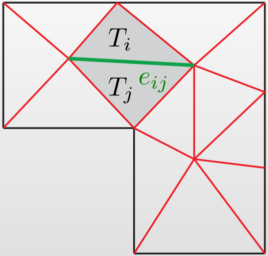
$$\begin{aligned} -\Delta u &= f, && \text{on } \Omega \\ u &= 0, && \text{on } \partial\Omega \end{aligned}$$

Assume $\Omega \subset \mathbb{R}^d$ is bounded, sufficiently regular domain.

Notation:

- $(u, v)_D := \int_D u\, v\, dx \quad \forall\, D \subseteq \Omega$
- $\|v\|_{0,D} := \sqrt{(v, v)_D}, \quad |v|_{1,D} := \sqrt{(\boldsymbol{\nabla} v, \boldsymbol{\nabla} v)_D}$
- We will abbreviate $\boldsymbol{H}^1(\Omega) := \left[H^1(\Omega)\right]^d$ etc.

Meshes and basic notation



Meshes: (hanging nodes also possible)

$$\mathcal{T}_h := \{\, T_i \,\}, \text{ where } \overline{\Omega} = \bigcup \overline{T_i}$$
$$\partial\mathcal{T}_h := \{\, \partial T_i \,\}$$
$$\mathcal{E}_h := \{\, e_{ij} \;:\; e_{ij} = \partial T_i \cap \partial T_j,\; i > j \,\}$$

Broken Sobolev spaces:

$$H^s(\mathcal{T}_h) := \{\, v \in L^2(\Omega) \;:\; v|_T \in H^s(T) \text{ for all } T \in \mathcal{T}_h \,\}$$
$$(u,v)_{\mathcal{T}_h} := \sum_{T\in\mathcal{T}_h} (u,v)_{L^2(T)}, \quad \|v\|_{\mathcal{T}_h} := (v,v)^{1/2}_{\mathcal{T}_h}, \quad \text{etc.}$$

Jump and average:

$$[\![v]\!] := v|_{T_i} - v|_{T_j}, \quad \{v\} := \tfrac{1}{2}(v|_{T_i} + v|_{T_j})$$

Note: functions defined on $\mathcal{E}_h$ can be interpreted as functions on $\partial\mathcal{T}_h$.

A simple DG method

Example: Symmetric Interior Penalty Galerkin (SIPG[1])

Find $u \in H^s(\mathcal{T}_h) \cap H^1_0(\Omega)$, $s > 3/2$, such that

$$a_h^{SIPG}(u,v) = (f,v)_{\mathcal{T}_h}, \qquad \forall\, v \in H^s(\mathcal{T}_h),$$

where we define a symmetric bilinear form

$$\begin{aligned} a_h^{SIPG}(u,v) :=& (\boldsymbol{\nabla} u, \boldsymbol{\nabla v})_{\mathcal{T}_h} - (\{\partial_n u\}, [\![v]\!])_{\mathcal{E}_h} \\ & - (\{\partial_n v\}, [\![u]\!])_{\mathcal{E}_h} + \tau([\![u]\!],[\![v]\!])_{\mathcal{E}_h}. \end{aligned}$$

Here, $\tau := \alpha \frac{p^2}{h}$ is a penalty parameter, with $\alpha \in \mathbb{R}^+$.

Natural energy norm for the discrete analysis:

$$\|v\|_{1,h} := \left(\|\boldsymbol{\nabla} v\|^2_{\mathcal{T}_h} + \|\tau^{1/2}[\![v]\!]\|^2_{\mathcal{E}_h} \right)^{1/2}.$$

[1] e.g. [Arnold 1982], [Rivière 2008]

Hybridized DG method

Example: Hybridized Symmetric Interior Penalty Galerkin

Find $(u, \hat{u}) \in (H^s(\mathcal{T}_h) \cap H^1_0(\Omega)) \times L^2(\mathcal{E}_h)$, $s > 3/2$, such that

$$a_h(u, \hat{u}; v, \hat{v}) = (f, v)_{\mathcal{T}_h}, \qquad \forall\, (v, \hat{v}) \in H^s(\mathcal{T}_h) \times L^2(\mathcal{E}_h),$$

where $\hat{u} := \{u\}$ and $\hat{v} := \{v\}$. We define a symmetric bilinear form

$$\begin{aligned} a_h(u, \hat{u}; v, \hat{v}) :=& (\boldsymbol{\nabla} u, \boldsymbol{\nabla} \boldsymbol{v})_{\mathcal{T}_h} - (\partial_n u, v - \hat{v})_{\partial\mathcal{T}_h} \\ & - (\partial_n v, u - \hat{u})_{\partial\mathcal{T}_h} + \tau(u - \hat{u}, v - \hat{v})_{\partial\mathcal{T}_h}. \end{aligned}$$

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Natural energy norm for the discrete analysis:

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Hybridization: From DG to HDG (4/4)

Given a triangulation $\mathcal{T}_h$, define finite dimensional spaces V_h and $\widehat{V}_h$:

$$V_h := \{ v \in H^1(\mathcal{T}_h) \ : \ v|_T \in \mathcal{P}^p(T),\ \forall\, T \in \mathcal{T}_h \}$$
$$\hat{V}_h := \{ \hat{v} \in L^2(\mathcal{E}_h) \ : \ v|_E \in \mathcal{P}^p(E),\ \forall\, E \in \mathcal{E}_h,\ \hat{v} = 0 \text{ on } \partial\Omega \}$$

Discrete problem: Find $(u_h, \hat{u}_h) \in V_h \times \widehat{V}_h$ such that

$$a_h(u_h, \hat{u}_h; v_h, \hat{v}_h) = (f, v_h)_\Omega, \qquad \forall (v_h, \hat{v}_h) \in V_h \times \widehat{V}_h.$$

Stability analysis: For any $\alpha > \alpha_0$, there holds[2]

$$a_h(v_h, \hat{v}_h; v_h, \hat{v}_h) \geq \tfrac{1}{2} \|(v_h, \hat{v}_h)\|^2_{1,h}, \qquad \forall (v_h, \hat{v}_h) \in V_h \times \widehat{V}_h,$$

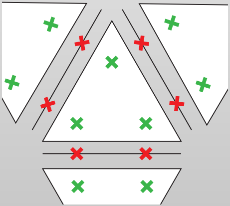
where α_0 depends on the element shape. One can choose α *'large enough'* (e.g. $\alpha = 10$) or explicitly compute $\alpha_0|_T$ on d-simplices and d-hypercubes.

[2] e.g. [Egger 2008]

Some remarks

- The hybridized method is consistent by construction!
- Standard error analysis yields optimal error estimates.
- Stability is independent of the particular choice of $\widehat{V}_h$.
- More general boundary conditions are possible.
- Other problems were also investigated, e.g.
  - Convection-diffusion [Egger and Schöberl 2009]
  - Stokes problem [Cockburn et. al. 2010], [Egger and W. 2010b]
  - Oseen problem (in preparation)

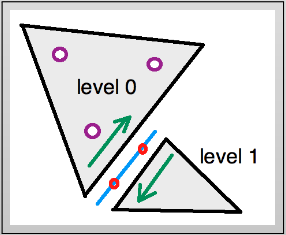
- Assembly in an element-wise fashion.
- Static condensation on element level.
- Upwinding can be easily incorporated.
- Locally varying polynomial degrees and nonconforming h-refinements possible.



HDG methods: Implementation (1/2)

Implementation in DUNE

- Implementation uses `dune-pdelab` and the core modules.
- Approximations in the interior of elements by monomials (using `MonomLocalFiniteElementMap`)
- Approximations on the element borders require some extra work:
  - We use the `IntersectionIndexSet` provided in PDELab and a self-written `FaceMonomLocalFiniteElementMap` to define a grid function space on the faces of the mesh.
  - **Problem:** The orientation of intersections may differ in two intersecting elements.

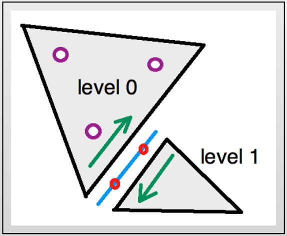


This causes problems when mapping from intersection coordinates to coordinates inside the element.

HDG methods (2/2)

Implementation in DUNE

- The only solution we know so far is a little helper that finds the corresponding intersection in the outside element if it has a lower index or a higher level than the current intersection.



- We then map the coordinates as follows:

```
if(wrongintersection)
    return rightintersection->geometryInOutside().global(x);
else
    return rightintersection->geometryInInside().global(x);
```

- The big disadvantage here is that we need to instantiate a quadratic number of intersections (performance issues?!)

Numerical example: Kovaznay (1/3)

Example: Oseen problem; $\Omega = (-0.5, 2) \times (-0.5, 1.5)$ [Kovasznay 1947]

$$\left.\begin{aligned} -\Delta\,\boldsymbol{u} + \boldsymbol{w}\nabla\boldsymbol{u} + \nabla p &= \boldsymbol{0} \\ \operatorname{div}\boldsymbol{u} &= 0 \end{aligned}\right\} \text{ on } \Omega, \qquad \boldsymbol{u} = \boldsymbol{u}_{exact} \text{ on } \partial\Omega$$

Exact solution:

$$\boldsymbol{u}(x,y) = \begin{pmatrix} 1 - \exp(\lambda x)\cos(2\pi y) \\ \frac{\lambda}{2\pi}\exp(\lambda x)\sin(2\pi y) \end{pmatrix},$$

$$p(x,y) = -\tfrac{1}{2}\exp(2\lambda x) + \overline{p},$$

with parameters λ and $\overline{p}$ given as

$$\lambda := \frac{-8\pi^2}{\nu^{-1} + \sqrt{\nu^{-2} + 16\pi^2}}$$

and $\overline{p} = 2\int_{-\frac{1}{2}}^{\frac{3}{2}} \exp(2\lambda x)\,dx$.

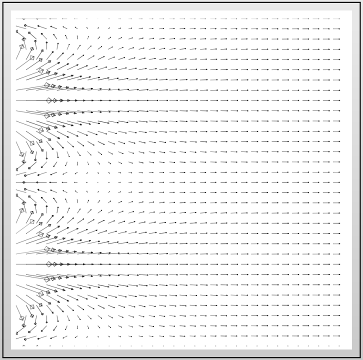


Figure: Velocity vectors ($\nu = 1/10$)

Numerical example: Kovaznay (2/3)

level	dimension		L^2 error		energy error	
$\mathbf{P_1 - P_0}$	K	S	error	rate	error	rate
0	578	326	37.5126	–	229.691	–
1	2248	1240	12.6979	1.56	143.706	0.68
2	8864	4832	3.70855	1.78	80.5438	0.84
3	35200	19072	0.97223	1.93	41.9258	0.94
4	140288	75766	0.24589	1.98	21.2533	0.98
$\mathbf{P_2 - P_1}$	K	S	error	rate	error	rate
0	1056	468	29.0005	–	217.573	–
1	4128	1776	4.34306	2.74	20.0395	3.44
2	16320	6912	0.63415	2.78	5.61554	1.84
3	64896	27264	0.08338	2.93	1.45829	1.95
4	258816	108288	0.01054	2.98	0.36827	1.99
$\mathbf{P_3 - P_2}$	K	S	error	rate	error	rate
0	1660	610	10.7941	–	93.9881	–
1	6512	2312	0.97279	3.47	14.7614	2.67
2	25792	8992	0.07140	3.77	2.08979	2.82
3	102656	34456	0.00461	3.95	0.26982	2.95
4	409600	140800	0.00029	3.98	0.03397	2.99

Table: **Kovsznay flow:** Errors of the numerical solution for different inf-sup stable finite element approximations and a sequence of uniformly refined meshes.

Numerical example: Kovaznay (3/3)

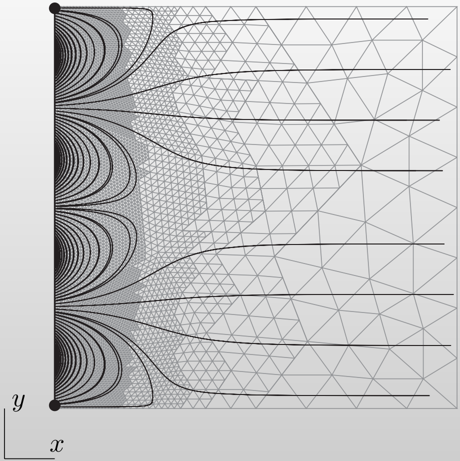


Figure: **Kovasznay flow** ($\nu = 1$): Streamlines and nonconforming mesh after 4 subsequent h-refinements.

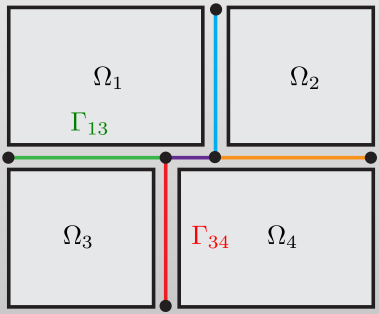
Outline

Hybridized mortar method (1/2)

Similarly as for the HDG method, we can define a **hybridized mortar method**[3]:

$$V_h := \{\, v \in H_0^1(\Omega_h) \;:\; v|_T \in \mathcal{P}^p(T),\ \forall\, T \in \mathcal{T}_h \,\}$$

$$\hat{V}_h := \{\, \hat{v} \in L^2(\Gamma_h) \;:\; v|_E \in \mathcal{P}^p(E),\ \forall\, E \in \mathcal{T}_h(\Gamma_h) \,\}$$



partition:

$$\Omega_h := \{\Omega_1, \Omega_2, \ldots, \Omega_N\}$$

interfaces:

$$\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$$

$$\Gamma_h := \{\Gamma_{ij}\}$$

skeleton:

$$\Gamma := \bigcup \Gamma_{ij}$$

[3] cf. [Egger 2008]

Hybridized mortar method (2/2)

Discrete problem: Find $(u_h, \hat{u}_h) \in V_h \times \widehat{V}_h$ such that

$$a_h(u_h, \hat{u}_h; v_h, \hat{v}_h) = (f, v_h)_\Omega, \qquad \forall (v_h, \hat{v}_h) \in V_h \times \widehat{V}_h.$$

where we define

$$\begin{aligned} a_h(u, \hat{u}, v, \hat{v}) :=& (\nabla u, \nabla v)_{\Omega_h} - (\partial_n u, v - \hat{v})_{\partial\Omega_h} \\ & - (\partial_n v, u - \hat{u})_{\partial\Omega_h} + \tau (u - \hat{u}, v - \hat{v})_{\partial\Omega_h}. \end{aligned}$$

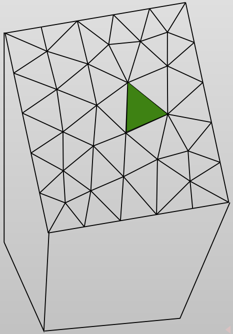
Important results:

- No direct coupling between subdomain solutions.
- *'Elimination'* on subdomains leads to a Schur complement system for the hybrid variables only ($\rightarrow$ domain decomposition methods).
- Space for the hybrid variable can be chosen with great flexibility (no inf-sup-condition necessary for multiplier).
- Hybridized mortar methods for other problems were also analyzed, e.g. Maxwell [Hollaus et. al. 2010], Stokes [Egger and W. 2010a].
- For the finest partition $\Omega_h = \mathcal{T}_h$, we recover the hybridized DG method.

Hybrid mortar: Implementation

Implementation in DUNE

- In many applications, there exists a parametrization for the interface (e.g. planar, cylindrical, spherical, ...)
- **Idea:** Lagrange multipliers live on $d-1$ dimensional structured meshes that are transformed to the physical space.
- Overlaps between multiplier mesh and subdomain meshes are computed using `dune-grid-glue` by C. Engwer and O. Sander.



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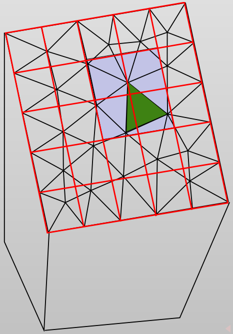

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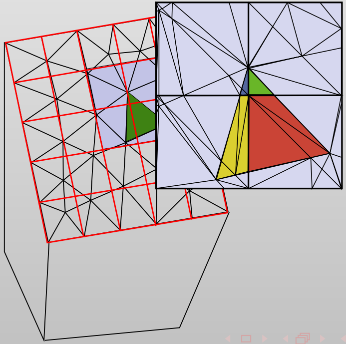
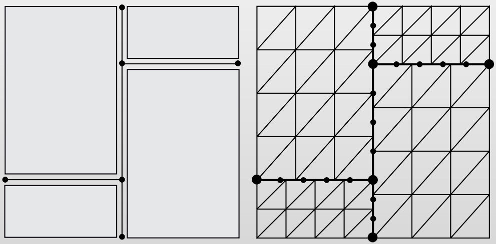


Hybrid mortar: Stokes problem

Stokes problem [Egger and W. 2010a]:

- Partitioning and triangulations:



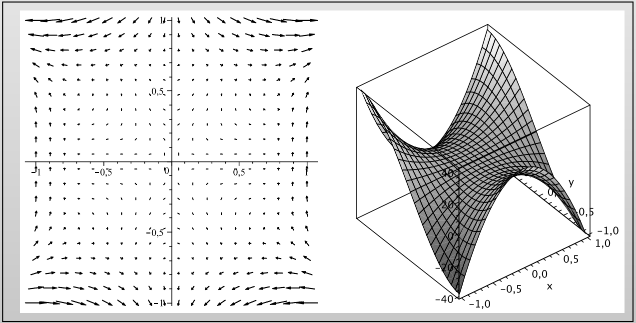
- The subdomain interfaces are extracted using a `Codim1Extractor`
- The entire `OneDGrid` is extracted with a `Codim0Extractor`
- A transformation is given to place the interface grids on the coupling boundaries.
- We use the `PSurface` backend to generate remote intersections.

Numerical example

Example: Stokes problem: Colliding flow, $\Omega = (-1,1)^2$

$$\left.\begin{array}{rcl} -\Delta\,\boldsymbol{u} + \nabla p & = & \mathbf{0} \\ \operatorname{div}\boldsymbol{u} & = & 0 \end{array}\right\} \text{ on } \Omega, \qquad \boldsymbol{u} = \boldsymbol{u}_{exact} \text{ on } \partial\Omega$$

$$\boldsymbol{u}_{exact} = \left(20xy^3, 5x^4 - 5y^4\right), \; p_{exact} = 60x^2y - 20y^3$$



Plot of analytic solution; velocity vectors and pressure field

Mortar: Numerical results (1/2)

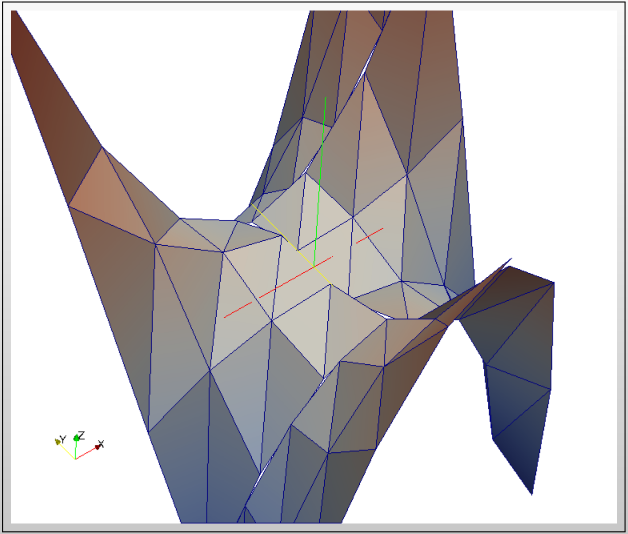


Figure: Numerical results (p=2): h: 1.00; L^2-error: 1.418; energy error: 3.253

Mortar: Numerical results (1/2)

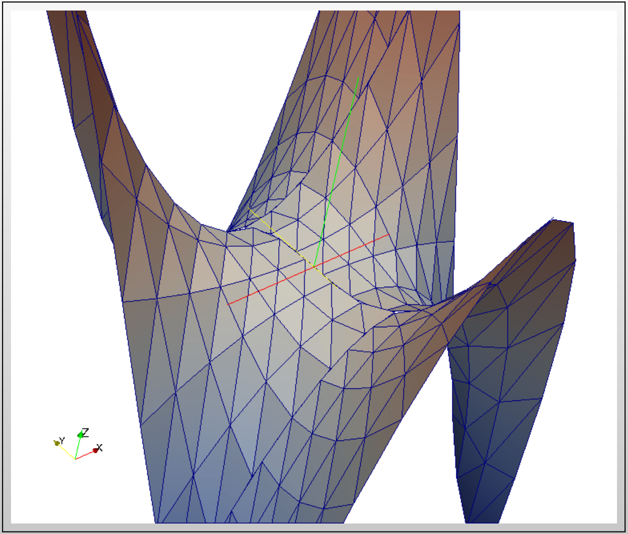


Figure: Numerical results (p=2): h: 0.50; L^2-error: 0.171; energy error: 0.761

Mortar: Numerical results (1/2)

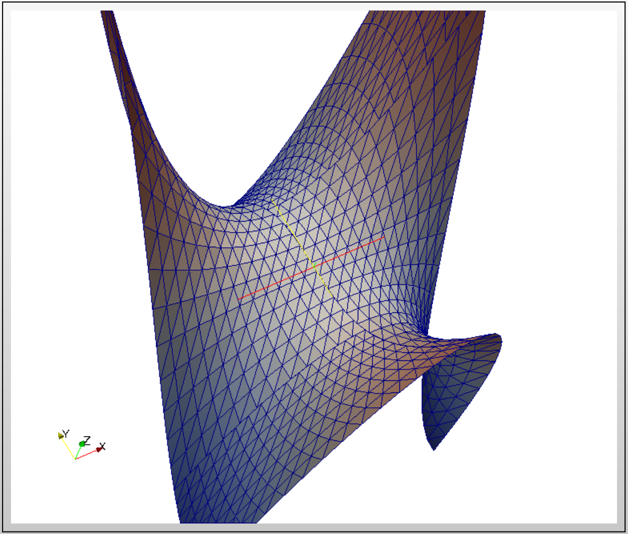


Figure: Numerical results (p=2): h: 0.25; L^2-error: 0.021; energy error: 0.183

Mortar: Numerical results (2/2)

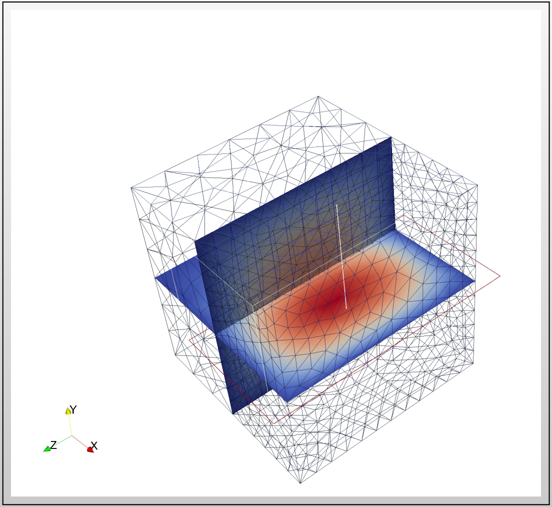


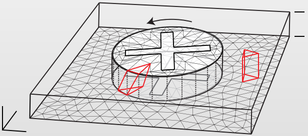
Figure: Simple example for a diffusion problem with non-matching meshes in 3D

Outline

Conclusion

Summary

- Presentation of hybridized DG methods.
- Implementation of HDG and hybridized mortar methods in DUNE.
- Possible applications include different interface problems (e.g. propellers).



Known issues

- HDG methods in DUNE not (yet) naturally implementable?

Future work

- Implementation of parallel codes for high performance computing.
- Analysis and implementation for time-dependent problems.
- Use of efficient domain decomposition solvers.

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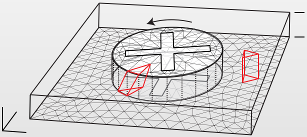

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Thanks for your attention!

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